Actuarial Mathematics Introduction

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Introduction

Actuarial science is the discipline that applies mathematical and statistical methods to assess risk in the insurance and finance industries. Actuaries are professionals who are qualified in this field through education and experience. In the United States, Canada, the United Kingdom, France and several other countries, actuaries must demonstrate their competence by passing a series of rigorous professional examinations.

Actuarial science includes a number of interrelating subjects, including probability, mathematics, statistics, finance, economics, and computer programming. Historically, actuarial science used deterministic models in the construction of tables and premiums. The science has gone through revolutionary changes during the last 30 years due to the proliferation of high speed computers and the synergy of stochastic actuarial models with modern financial theory.

Many universities have undergraduate and graduate degree programs in actuarial science. In 2010, a study published by job search website CareerCast ranked actuary as the first job in the United States. A similar study by U.S. News & World Report in 2006 included actuaries among the 25 Best Professions that it expects will be in great demand in the future.

This course will focus on insurance and reinsurance actuarial mathematics theories and their applications in terms of implementation.
First, we present some generalities. Then we present life and non-life insurance mathematics. Finally, we show the application of actuarial mathematics in the reinsurance field. We illustrate the theory with several examples using Excel and Visual Basic for Applications available in Excel, when an implementation is needed.

1 Generalities

1.1 History

Need for insurance

The basic requirements of communal interests gave rise to risk sharing since the dawn of civilization. For example, people who lived their entire lives in a camp had the risk of fire, which would leave their band or family without shelter. After basic exchange came into existence, more complex forms developed beyond a basic barter economy, and new forms of risk manifested. Merchants embarking on trade journeys bore the risk of losing goods entrusted to them, their own possessions, or even their lives. Intermediaries developed warehouse and trade goods, and they often suffered from financial risk. The primary providers in any extended families or households always ran the risk of premature death, disability or infirmity, leaving their dependents to starve. Credit procurement was difficult if the lender worried about repayment in the event of the borrower's death or infirmity. Alternatively, people sometimes lived too long, exhausting their savings, if any, or becoming a burden on others in the extended family or society.

Early attempts

In the ancient world there was not always room for the sick, suffering, disabled, aged, or the poor—these were often not part of the cultural consciousness of societies. Early methods of protection, aside from the normal support of the extended family, involved charity; religious organizations or neighbors would collect for the destitute and needy. By the middle of the third century, 1,500 suffering people were being supported by charitable operations in Rome. Charitable protection is still an active form of support to this very day. However, receiving charity is uncertain and is often accompanied by social stigma. Elementary mutual aid agreements and pensions did arise in antiquity. Early in the Roman Empire, associations were formed to meet the expenses of burial, cremation, and monuments—precursors to burial insurance and friendly societies. A small sum was paid into a communal fund on a weekly basis, and upon the death of a member, the fund would cover the expenses of rites and burial. These societies sometimes sold shares in the building of columbaria, or burial vaults, owned by the fund—the precursor to mutual insurance companies. Other early examples of mutual surety and assurance pacts can be traced back to various forms of fellowship within the Saxon clans of England and their Germanic forbears, and to Celtic society. However, many of these earlier forms of surety and aid would fail due to lack of understanding and knowledge.

Development of theory

The 17th century was a period of extraordinary advances in mathematics in Germany, France, and England. At the same time there was a rapidly growing desire and need to place the valuation of personal risk on a more scientific basis. Independently from each other, compound interest was studied and probability theory emerged as a well understood mathematical discipline. Another important advance came in 1662 from a London draper named John Graunt, who showed that there were predictable patterns of longevity and death in a defined group, or cohort, of people, despite the uncertainty about the future longevity or mortality of any one individual person. This study became the basis for the original life table (see also Appendix B Mortality Tables). It was now possible to set up an insurance scheme to provide life insurance or pensions for a group of people, and to calculate with some degree of accuracy how much each person in the group should contribute to a common fund assumed to earn a fixed rate of interest. The first person to demonstrate publicly how this could be done was Edmond Halley. In addition to constructing his own life table, Halley demonstrated a method of using his life table to calculate the premium someone of a given age should pay to purchase a life-annuity.

Early actuaries

James Dodson's pioneering work on the level premium system led to the formation of the Society for Equitable Assurances on Lives and Survivorship (now commonly known as Equitable Life) in London in 1762. This was the first life insurance company to use premium rates which were calculated scientifically for long-term life policies, using Dodson's work. The company still exists, though it has run into difficulties recently. After Dodson's death in 1757, Edward Rowe Mores took over the leadership of the group that eventually became the Society for Equitable Assurances in 1762. It was he who specified that the chief official should be called an '**actuary**'. Previously, the use of the term had been restricted to an official who recorded the decisions, or 'acts', of ecclesiastical courts, in ancient times originally the secretary of the Roman senate, responsible for compiling the Acta Senatus. Other companies which did not originally use such mathematical and scientific methods most often failed or were forced to adopt the methods pioneered by Equitable.

Development of the modern profession

In the eighteenth and nineteenth centuries, computational complexity was limited to manual calculations. The actual calculations required to compute fair insurance premiums are rather complex. The actuaries of that time developed methods to construct easily-used tables, using sophisticated approximations called commutation functions, to facilitate timely, accurate, manual calculations of premiums. Over time, actuarial organizations were founded to support and further both actuaries and actuarial science, and to protect the public interest by ensuring competency and ethical standards. However, calculations remained cumbersome, and actuarial shortcuts were commonplace. Non-life actuaries followed in the footsteps of their life compatriots in the early twentieth century. In the United States, the 1920 revision to workers' compensation rates took over two months of around-the-clock work by day and night teams of actuaries. In the 1930s and 1940s, however, rigorous mathematical foundations for stochastic processes were developed. Actuaries could now begin to forecast losses using models of random events instead of deterministic methods. Computers further revolutionized the actuarial profession. From pencil-and-paper to punchcard to microcomputers, the modeling and forecasting ability of the actuary has grown exponentially.

Another modern development is the convergence of modern financial theory with actuarial science. In the early twentieth century, actuaries were developing many techniques that can be found in modern financial theory, but for various historical reasons, these developments did not achieve much recognition. However, in the late 1980s and early 1990s, there was a distinct effort for actuaries to combine financial theory and stochastic methods into their established models. Today, the profession, both in practice and in the educational syllabi of many actuarial organizations, combines tables, loss models, stochastic methods, and financial theory, but is still not completely aligned with modern financial economics.

Year	Comment
1662	First mortality table (John Graunt)
1693	Mortality table of Edmund Halley
1725	"Annuities on Lives" by Abraham de Moivre (London)
1755	"Mathematical Repository" by James Dodson (London)
1756	Table of annual level premiums for a whole life assurance by James Dodson; analysis of their management
1762	Foundation of the Society for Equitable Assurances on Lives and Survivorships
1671 and 1693	Formulae for the appraisal of annuities of Jan de Witt and Edmund Halley
1774	"Gambling Act" (or "Life Assurance Act"), King George III

Figure 1: History

1.2 Insurance

Insurance involves pooling funds from many insured entities (known as exposures) in order to pay for relatively uncommon but severely devastating losses which can occur to these entities. The insured entities are therefore protected from risk for a fee, with the fee being dependent upon the frequency and severity of the event occurring. In order to be insurable, the risk insured against must meet certain characteristics in order to be an insurable risk. Insurance is a commercial enterprise and a major part of the financial services industry, but individual entities can also self-insure through saving money for possible future losses.

Insurance companies may be classified into two groups:

- Life insurance companies, which sell life insurance, annuities and pensions products.
- Non-life, General, or Property/Casualty insurance companies, which sell other types of insurance.

In most countries, life and non-life insurers are subject to different regulatory regimes and different tax and accounting rules. The main reason for the distinction between the two types of company is that life, annuity, and pension business is very long-term in nature — coverage for life assurance or a pension can cover risks over many decades. By contrast, non-life insurance cover usually covers a shorter period, such as one year.

Insurance companies are generally classified as either mutual or stock companies. Mutual companies are owned by the policyholders, while stockholders (who may or may not own policies) own stock insurance companies. Demutualization of mutual insurers to form stock companies, as well as the formation of a hybrid known as a mutual holding company, became common in some countries.

Other possible forms for an insurance company include reciprocals, in which policyholders 'reciprocate' in sharing risks, and Lloyd's organizations.

Insurance companies are rated by various agencies such as A. M. Best. The ratings include the company's financial strength, which measures its ability to pay claims. It also rates financial instruments issued by the insurance company, such as bonds, notes, and securitization products.

Reinsurance companies are insurance companies that sell policies to other insurance companies, allowing them to reduce their risks and protect themselves from very large losses. The reinsurance market is dominated by a few very large companies, with huge reserves. A reinsurer may also be a direct writer of insurance risks as well.

The financial stability and strength of an insurance company should be a major consideration when buying an insurance contract. An insurance premium paid currently provides coverage for losses that might arise many years in the future. For that reason, the viability of the insurance carrier is very important. In recent years, a number of insurance companies have become insolvent, leaving their policyholders with no coverage (or coverage only from a government-backed insurance pool or other arrangement with less attractive payouts for losses). A number of independent rating agencies provide information and rate the financial viability of insurance companies.

<table>
<tr><th colspan="2">Insurance</th></tr>
<tr><td>General insurance
(damages or civil responsibility)</td><td>Life insurance</td></tr>
<tr><td>If a claim occurs, the insurer covers its (random) cost.
Usually short-term (1 year at most).</td><td>The amount insured is given.
medium or long-term.</td></tr>
<tr><td colspan="2">Types</td></tr>
<tr><td>– fire insurance
– theft insurance
– sickness and disability insurance
– Civil Responsibility insurance
(motor, of the manager, products, professional, family)
– motor insurance
– credit insurance
– natural disaster insurance
– transport insurance
– flight insurance
– “technical” insurance
– assistance</td><td>– insurances on the duration of life
– insurance for wedding and births
– insurances linked to investment funds
– financial operation
– disability annuities
– pension fund management
– ancillary benefit</td></tr>
</table>

<table>
<tr><th colspan="2">Insurance</th></tr>
<tr><td>Insurances of the person</td><td>Group insurance</td></tr>
<tr><td>– needs emerging from random situations:
duration of life → insurances on the duration of life
changes in the normal health status→ sickness and disability, insurance, disability annuities (“health insurances”)</td><td>– with living or death benefits
– pension funds
– social insurance</td></tr>
</table>

Figure 2: Types of Insurance

2 Life Insurance Mathematics

Life insurance is a contract between the policy owner and the insurer, where the insurer agrees to pay a designated beneficiary a sum of money upon the occurrence of the insured individual's or individuals' death or other event, such as terminal illness or critical illness. In return, the policy owner agrees to pay a stipulated amount (at regular intervals or in lump sums).
Hence the value for the policyholder is derived from an **uncertain future** claim event that could occur during its live.

Life policies are legal contracts and the terms of the contract describe the limitations of the insured events. Specific exclusions are often written into the contract to limit the liability of the insurer; for example claims relating to suicide, fraud, war, riot and civil commotion.

Life-based contracts tend to fall into two major categories:

- Protection policies - designed to provide a benefit in the event of specified event, typically a lump sum payment. A common form of this design is term insurance.
- Investment policies - where the main objective is to facilitate the growth of capital by regular or single premiums. Common forms are whole life, universal life and variable life policies.

In the following section, we focus more on the insurances on the duration of life. We present one of the most important determinants of these insurances: the demographical basis. Then, we show how some of these insurances are evaluated with Life Insurance mathematics.

2.1 Basic concepts

2.1.1 Certain Present Value

If the payment period of an annuity is independent of any life event, this is known as an **annuity-certain**. It is different from a life annuity (payments end on the beneficiary's death)

The Present Value of an annuity, which is a series of unit payments at the end of each year for n years is obtained from:

$$PV\ (\text{unit annuity} - \text{certain}) = \sum_{k=1}^{n} \frac{1}{(1+i)^k}$$

i is the annual interest rate.

With $\boldsymbol{v} := \frac{1}{1+i}$ we get:

$$PV\ (\text{unit annuity} - \text{certain}) = \sum_{k=1}^{n} v^k$$

The Present Value of an annuity, which is a series of unit payments at the beginning of each year for n years (end at the beginning of year n) is obtained from:

$$\ddot{PV}\ (\text{unit annuity} - \text{certain}) = \sum_{k=0}^{n} \frac{1}{(1+i)^k} = 1 + PV\ (\text{unit annuity} - \text{certain})$$

2.1.2 Mortality Table

A mortality table (also called a life table or actuarial table) gives the average numbers $\boldsymbol{l_x}$ of survivors at the various ages in a given group.
More precisely, reference is made to a group of l_α persons with the same age (α), closed to new entries ("**cohort**"), the only cause of exit being death and with durations of life identically distributed.

Given (α) and a proper "high age" (ω) → "maximum age" such that $l_\omega \cong 0$ we define mortality table the sequence:

$$l_\alpha, l_{\alpha+1}, \dots, l_x, l_{x+1}, \dots, l_\omega$$

(x) is an integer number and l_α represents the "root" of the table. Usually $\alpha = 0$ and $l_\alpha = 100\,000$.

Hence, a mortality table shows also, for each age (x), what the probability is that a person of that age will die before his next birthday. From this starting point, a number of statistics can be derived and thus also included in the table:

- the probability of surviving any particular year of age,
- remaining life expectancy for people at different ages,
- the proportion of the original birth cohort still alive,
- estimates of a cohort's longevity characteristics.

Life tables are usually constructed separately for men and for women because of their substantially different mortality rates. Other characteristics can also be used to distinguish different risks, such as smoking status, occupation, and socio-economic class.

Life tables can be extended to include other information in addition to mortality, for instance health information to calculate health expectancy. Health expectancies, of which disability-free life expectancy (DFLE) and Healthy Life Years (HLY) are the best-known examples, are the remaining number of years a person can expect to live in a specific health state, such as free of disability. Two types of life tables are used to divide the life expectancy into life spent in various states: 1) multi-state life tables (also known as increment-decrement life tables) based on transition rates in and out of the different states and to death, and 2) prevalence-based life tables (also known as the Sullivan method) based on external information on the proportion in each state. Life tables can also be extended to show life expectancies in different labor force states or marital status states.

The Appendix B Mortality Tables shows some examples of French mortality tables.

The basic algebra used in life tables is as follows.

- $\boldsymbol{q_x}$: the probability that someone aged (x) will die before reaching age (x+1)

- $\boldsymbol{p_x}$: the probability that someone aged (x) will survive to age (x+1).

Equation 1: Probability principle

$$p_x + q_x = 1$$

- Furthermore, we have:

$$l_{x+1} = l_x * (1 - q_x)$$

So,

Equation 2: Death probability

$$q_x = \frac{l_x - l_{x+1}}{l_x}$$

And,

$$l_{x+1} = l_x * p_x$$

$$p_x = \frac{l_{x+1}}{l_x}$$

- $\boldsymbol{d_x}$: the number of people who die aged (x).

$$d_x = l_x - l_{x+1} = l_x * (1 - p_x) = l_x * q_x$$

These symbols also extend to multiple years, by adding the number of years at the bottom left of the basic symbol.

- $\boldsymbol{{}_nd_x}$: the number of people who die between age x and age x + n.

$${}_nd_x = d_x + d_{x+1} + \cdots + d_{x+n-1} = l_x - l_{x+n}$$

- $_n\boldsymbol{q}_x$**:** the probability of death between the ages of x and age x + n.

$$ {}_nq_x = \frac{{}_nd_x}{l_x} = \frac{l_x - l_{x+n}}{l_x} $$

- $_n\boldsymbol{p}_x$**:** the probability that someone aged (x) will survive for (n) more years, i.e. live up to at least age (x+n) years.

Equation 3: Survival

$$ {}_np_x = \frac{l_{x+n}}{l_x} $$

- $_{m|n}\boldsymbol{q}_x$**:** the probability that someone aged (x) will survive for (m) more years, then die within the following (n) years.

$$ {}_{m|n}q_x = {}_mp_x * {}_nq_{x+m} = \frac{l_{x+m} - l_{x+m+n}}{l_x} $$

- $\boldsymbol{e}_x$: the expectation of life for the people alive at age x. This is the expected number of years remaining to live.

$$ e_x = \sum_{k=1}^{\omega - x} {}_kp_x = \sum_{k=1}^{\omega - x} \frac{l_{x+k}}{l_x} $$

Note1: For the implemented examples we assume that we are **in France before the 20/12/2005**. Indeed, there were some amendments about the tables (see Article A335-1 of the French Insurance Code) that we don't focus on for the purposes of this two day introduction in actuarial mathematics.

Note 2: If a unique table (same for men and women) is used, the prudential principle has to be applied (see Article A335-1 of the French Insurance Code for more details) to determine which one has to be used for calculations → generally: **TD88-90 / TV88-90** for insurance in case of **death / life**.

2.1.3 Actuarial Present Value

In actuarial science, the actuarial value or actuarial present value of a payment or series of payments which are random variables is the expected value of the present value of the payments, or equivalently, the present value of their expected values.

Actuarial present values are calculated for the payment or series of payments associated with **a discount factor and a probability**. In this case, the probability of a future payment is based on assumptions about the person's future mortality taking into account the person's age and an assumed life table, while the present value of those future assumed payments depend upon the interest rate used to discount them for the passage of time

2.2 Insurances in case of life

2.2.1 Pure endowment

A pure endowment is a life insurance contract designed to pay a lump sum after a specified term (on its 'maturity').

The characteristics of the contracts are:
- 0 → issue time of the policy: 0,
- A → benefit: (capital) at time n (n given), if the insured (aged x at time 0) is alive.

Here and in the following we assumed that the benefit is a unitary amount (here A = 1).

Actuarial value (at time 0 for a unitary amount):

Equation 4: Pure Endowment

$$ {}_nE_x = \frac{1}{(1+i)^n} * \frac{l_{x+n}}{l_x} $$

So

$$ {}_nE_x = v^n * {}_np_x $$

2.2.2 Life annuities

A life annuity is a contract that pays periodically a certain amount up to the death of the annuitant. Generally speaking, an annuitant buys a life annuity and makes installment payments for it throughout his/her working life. Following retirement, the annuitant begins to receive the benefit, the amount of which may or may not be fixed in the annuity contract. A life annuity is designed to provide a stable income for the annuitant in retirement.

Actuarial value (at time 0 for a unitary amount):

Equation 5: Life annuity

If the amount is given at the beginning of each year up to the death:

$$ \ddot{a}_x = \sum_{k=0}^{\omega-x} \frac{1}{(1+i)^k} * \frac{l_{x+k}}{l_x} $$

So

$$ \ddot{a}_x = \sum_{k=0}^{\omega-x} {}_kE_x $$

If the amount is given at the end of each year up to the death:

$$ a_x = \sum_{k=1}^{\omega-x} {}_kE_x $$

If the amount is given at the beginning of each year up to the death but, for (n) years at most:

$$\ddot{a}_{x:n} = \sum_{k=0}^{n-1} \frac{1}{(1+i)^k} * \frac{l_{x+k}}{l_x}$$

So

$$\ddot{a}_{x:n} = \sum_{k=0}^{n-1} {}_kE_x$$

If the amount is given at the end of each year up to the death but, for (n) years at most:

$$a_{x:n} = \sum_{k=1}^{n} {}_kE_x$$

If the amount is given at the beginning of each year up to the death but, with deferment m years:

$${}_m\ddot{a}_x = \sum_{k=m}^{\omega-x} \frac{1}{(1+i)^k} * \frac{l_{x+k}}{l_x}$$

So

$${}_m\ddot{a}_x = \sum_{k=m}^{\omega-x} {}_kE_x$$

If the amount is given at the end of each year up to the death but, with deferment m years:

$${}_m a_x = \sum_{k=m+1}^{\omega-x} {}_kE_x$$

If the amount is given at the beginning of each year up to the death but, for (n) years at most with deferment m years:

$${}_m\ddot{a}_{x:n} = \sum_{k=m}^{m+n-1} \frac{1}{(1+i)^k} * \frac{l_{x+k}}{l_x}$$

So

$${}_m\ddot{a}_{x:n} = \sum_{k=m}^{m+n-1} {}_kE_x$$

If the amount is given at the end of each year up to the death but, for (n) years at most with deferment m years:

$${}_m a_{x:n} = \sum_{k=m+1}^{m+n} {}_kE_x$$

2.3 Insurances in case of death

2.3.1 "Whole life" insurance

A whole life insurance is a life insurance policy that remains in force for the insured's whole life and paid a capital at the end of the year of the insured's death, whenever this occurs.

Actuarial value (at time 0 for a unitary amount):

$$A_x = \sum_{k=0}^{\omega-x-1} v^{k+1} * {}_{k|1}q_x$$

So

Equation 6: Whole Life

$$A_x = \sum_{k=0}^{\omega-x-1} {}_{k|1}A_x$$

When ${}_{k|1}\boldsymbol{A}_x = v^{k+1} * {}_{k|1}q_x$.

<u>Note</u>: A common assumption is that the date of the death is known at the middle of the year so that the capital is paid at this date. In this case $v^{k+1}$is replaced by $v^{k+\frac{1}{2}}$.

2.3.2 Term insurance

A term insurance is a life insurance policy that remains in force for a specified term of years and paid a capital at the end of the year of the insured's death, if it occurs before the end of the term period.

Actuarial value (at time 0 for a unitary amount):

$${}_nA_x = \sum_{k=0}^{n-1} v^{k+1} * {}_{k|1}q_x$$

So

Equation 7: Term Insurance

$${}_nA_x = \sum_{k=0}^{n-1} {}_{k|1}A_x$$

2.4 Insurances both in case of life and death

2.4.1 Endowment

An endowment policy is a life insurance contract designed to pay a lump sum after a specified term (on its 'maturity') or on earlier death. It is a combination of a pure endowment and a term insurance with the same payable amount in case of death and of survival.

Actuarial value (at time 0 for a unitary amount):

Equation 8: Endowment

$$A_{x:\overline{n}|} = {}_nE_x + {}_nA_x$$

2.4.2 Combined Endowment

A combined endowment is an endowment with an amount in case of survival different from the death benefit.

Actuarial value (at time 0 for a unitary amount in case of death):

$$A_{x:\overline{n}|} = (1+k) * {}_nE_x + {}_nA_x = A_{x:\overline{n}|} + k * {}_nE_x$$

With $k > -1$.

More actuarial formulae are reported in Appendix A Actuarial formulae.

2.5 Premiums

2.5.1 Principle of premium calculation

The principle of premium calculation in life insurance is the equity principle.
Let

- Y : random present value of benefits
- **U**: single premium, to be paid at policy issue (time 0),

Then, the random loss is:

$$L = Y - U$$

And from the equivalence principle (or actuarial equilibrium principle) we have:

$$E(L) = 0$$

So

Equation 9: Single Premium

$$U = E(Y)$$

U is the net single equivalence premium.

Life Insurance contract	Single Premium
Pure Endowment	$U = {}_nE_x$
Life Annuity (payable at the end of the year)	$U = a_x$
Deferred life annuity (payable at the beginning of the year for n years at most)	$U = {}_{m'}\ddot{a}_{x:n}$
Deferred life annuity (payable at the beginning of the year up to the death)	$U = {}_{m'}\ddot{a}_x$
Whole Life	$U = A_x$
Term Insurance	$U = {}_nA_x$
Endowment	$U = A_{x:\overline{n}}$

Theoretically, the expected loss is equal to the expected profit that is equal to zero as shown above. However, in the insurance field, the premium contains an **implicit loading** (so profit), obtained through a proper technical basis; hence, the expected profit is positive.

Stress the table (**stress test**) allow to analyze extremes situations or implicit loadings. Stress tests are linked to the sensitivities (sensitivities: few changes v/s Stress tests: a huge variation).

Standard Mortality Table → prudential → Demography bases of 1st order → fair value
Modified Mortality Table → realistic → Demography bases of 2nd order → realistic value

$$\text{Implicit Loadings} = \text{Fair Price} - \text{Realistic Price}$$

2.5.2 Annual level premiums

Annual level premium is a payment that is constant from year to year. The premium may be paid throughout the life of an insured or may be limited to a maximum number, such as 30 annual premiums. The premium is based on interest and a mortality assumption.

Let

- Y: the random present value of benefits
- **P**: the annual level premium,
- **X**: the random present value of the premiums, to be paid at policy issue (time 0),

Then, the random loss is:

$$L = Y - X$$

And from the equivalence principle (or actuarial equilibrium principle) we have E (income) = E (outcome):

$$E(X) = E(Y)$$

So,

$$E(X) = U$$

For example, let's consider an n year contract. If the annual level premium is payable for m ($m \leq n$) years at the beginning of the year, then:

$$E(X) = \sum_{k=0}^{m-1} P * \frac{1}{(1+i)^k} * \frac{l_{x+k}}{l_x}$$

So,

$$P = \frac{U}{\ddot{a}_{x:m}}$$

For annual level premium payable at the beginning of the year we have:

Life Insurance contract	**Premiums period**	**Annual level premium**
Pure Endowment	$m\ (m \leq n)$	$P = \frac{{}_{n}^{\blacksquare}E_x}{\ddot{a}_{x:m}}$
Deferred life annuity (payable at the beginning of the year for n years at most)	$m\ (m \leq m')$	$P = \frac{{}_{m'}^{\blacksquare}\ddot{a}_{x:n}}{\ddot{a}_{x:m}}$
Deferred life annuity (payable at the beginning of the year up to the death)	$m\ (m \leq m')$	$P = \frac{{}_{m'}^{\blacksquare}\ddot{a}_x}{\ddot{a}_{x:m}}$
Whole Life	up to death	$P = \frac{A_x}{\ddot{a}_x}$
Whole Life	up to time m	$P = \frac{A_x}{\ddot{a}_{x:m}}$
Term Insurance	$m\ (m \leq n)$	$P = \frac{{}_{n}^{\blacksquare}A_x}{\ddot{a}_{x:m}}$
Endowment	$m\ (m \leq n)$	$P = \frac{A_{x:\overline{n}}}{\ddot{a}_{x:m}}$

2.5.3 Natural premiums

The natural premiums are the premiums we have to pay each year and that correspond to a coverage of one year. So, for each year, there are equal to the expected annual costs. Hence to be cover for several years with natural premiums, the insured have to pay each year these annual premiums to be cover for the year in cure.
Hence, the level annual premium (constant) is payable throughout the policy duration and is an average of natural premiums (variable).

Let use the notation $\boldsymbol{P}_{.}^{(N)}$ for the natural premium.

The natural premiums of a pure endowment (for a unitary amount) are:

$$P_{h+1}^{(N)} = 0 \quad for \;\; h = 0, 1, \dots, n-2$$
$$P_{n}^{(N)} = {}_{1}E_{x+n-1}$$

The natural premiums of a term insurance (for a unitary amount) are:

$$P_{h+1}^{(N)} = {}_{1}A_{x+h} \quad for \;\; h = 0, 1, \dots, n-1$$

The natural premiums of an endowment (for a unitary amount) are:

$$P_{h+1}^{(N)} = {}_{1}A_{x+h} \quad for \;\; h = 0, 1, \dots, n-2$$
$$P_{n}^{(N)} = v$$

The natural premiums of a deferred life annuity (annual amount at the beginning of the year and for a unitary amount) are:

$$P_{h+1}^{(N)} = 0 \quad for \;\; h = 0, 1, \dots, n-1$$
$$P_{h+1}^{(N)} = 1 \quad for \;\; h = n, n+1, \dots$$

<u>Example</u>:

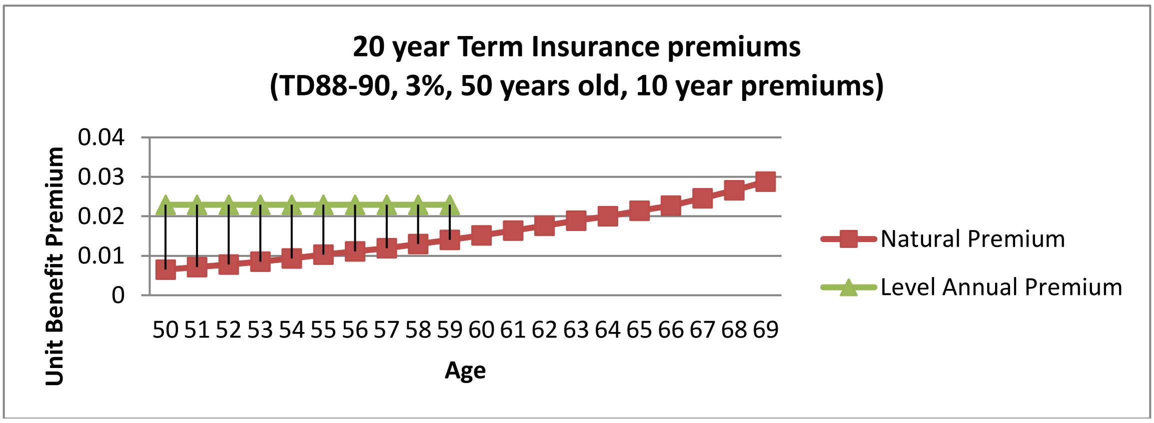


Figure 3: Natural and Level Annual Premiums

Figure 3: Natural and Level Annual Premiums show the two main periods of the insurance contact (with a level premium) displayed: during the first period, the insurer receives more premiums than the annual expected claims and less (nothing in the case shown) during the second period.
Therefore, in the business of insurance, an insurance company is legally required to maintain on its balance sheet some amounts with respect to the unmatured obligations (i.e., expected future claims).

2.6 Reserves

The reserves are the amount of funds necessary for a company to have at any given time to enable it, with interest and premiums paid as they shall occur, to meet all claims on the insurance then in force as they would mature according to the particular mortality table accepted.
The reserve is always reckoned as a liability, and is calculated on net premiums. It is theoretically the difference between the present value of the total insurance and the present value of the future premiums on the insurance.

2.6.1 Prospective Reserves

With the notations:
$Prest(t',t'')$ = actuarial value at time t' of benefits payable in (t',t''),
$Prem(t',t'')$ = actuarial value at time t' of premiums payable in (t',t'').
t' and t'' are integer numbers, i.e. policy anniversaries.

From the equivalence principle it follows

$$Prest(0,n) = Prem(0,n)$$

Where n is the maturity of the contract.

In general,

$$Prest(t,n) \neq Prem(t,n)$$
$$Prest(0,t) \neq Prem(0,t)$$

Where $t \in]0,n[$.

Usually,

$$Prest(t,t+1) \neq Prem(t,t+1)$$
i.e. annual expected cost $\neq$ annual premium.

With reference to the interval (t,n) let

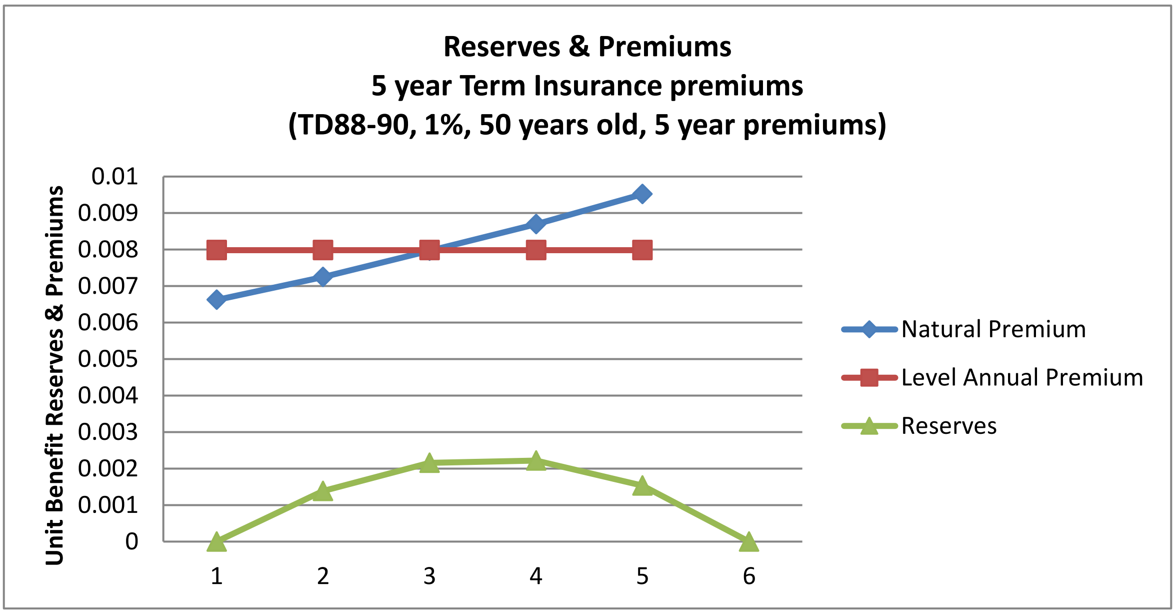
$$Prest(t,n) = Prem(t,n) + V_t$$

Where V_t allows to obtain the actuarial equilibrium at time t. Such quantity is called **prospective reserve**. So,

$$V_t = Prest(t,n) - Prem(t,n)$$

Examples:

Life Insurance contract	Reserves
Pure Endowment, annual premiums payable up to time n	$V_t = {}_{n-t}\blacksquare E_{x+t} - P * \ddot{a}_{x+t:n-t}$
Deferred life annuity (deferment m years, annual premiums up to time m payable at the beginning of the year)	$V_t = \begin{cases} {}_{m-t}\blacksquare\ddot{a}_{x+t} - P * \ddot{a}_{x+t:m-t} \; if\ t < m \\ \ddot{a}_{x+t} \; if\ t \geq m \end{cases}$
Whole Life, annual premiums payable up to death	$V_t = A_{x+t} - P * \ddot{a}_{x+t}$
Whole Life, annual premiums payable up to time m	$V_t = \begin{cases} A_{x+t} - P * \ddot{a}_{x+t:m-t} \; if\ t < m \\ A_{x+t} \; if\ t \geq m \end{cases}$
Term Insurance, annual premiums payable up to time n	$V_t = {}_{n-t}\blacksquare A_{x+t} - P * \ddot{a}_{x+t:n-t}$
Endowment, annual premiums payable up to time n	$V_t = A_{x+t:\overline{n-t}} - P * \ddot{a}_{x+t:n-t}$



2.6.2 Recurrent equations for the Reserves

Let us consider a generic policy with:

- annual premiums: $P_1, P_2, \dots P_n$,
- sum insured in case of death: $C_1, C_2, \dots C_n$,
- amount in case of survival: S,

Then, prospective reserves at time t are:

$$V_t = Prest(t,n) - Prem(t,n)$$

$$= \sum_{h=0}^{n-t-1} C_{t+h+1} * {}_{h|1}A_{x+t} + S * {}_{n-t}E_{x+t} - \sum_{h=0}^{n-t-1} P_{t+h+1} * {}_{h}E_{x+t}$$

$$= C_{t+1} * {}_{1}A_{x+t} - P_{t+1} + \sum_{h=1}^{n-t-1} C_{t+h+1} * {}_{h|1}A_{x+t} + S * {}_{n-t}E_{x+t} - \sum_{h=1}^{n-t-1} P_{t+h+1} * {}_{h}E_{x+t}$$

$$= C_{t+1} * {}_{1}A_{x+t} - P_{t+1} + V_{t+1} * {}_{1}E_{x+t}$$

Therefore

$$V_t + P_{t+1} = C_{t+1} * {}_{1}A_{x+t} + V_{t+1} * {}_{1}E_{x+t}$$

Or also

$$V_t + P_{t+1} = C_{t+1} * v * q_{x+t} + V_{t+1} * v * p_{x+t}$$

And

Equation 10: Recurrent equation for the Reserves

$$V_{t+1} = \frac{(V_t + P_{t+1}) - C_{t+1} * v * q_{x+t}}{v * p_{x+t}}$$

Note: As the equation refers to the interval $(t, t+1)$, it means that the equation: financial resources available (in year t) = liabilities (in year t) is a local actuarial equilibrium.

3 Non-Life Insurance Mathematics

Non-life insurance, also called property and casualty insurance, is a type of coverage that is very common and covers businesses and individuals. It protects them, monetarily, from disaster by providing money in the event of a financial loss.

Non-Life Insurance (such as home, auto, theft or damage, general liability) insurance covers something else other than a person's life.

3.1 Basic concepts

A basic concept of the theoretical approach is the **average cost** of the claims.

Let us consider:

- the insured population size: N,
- the number of claims in the population: n,
- the average cost or unit cost of the claims: $\bar{c}$,
- the total of the claims: C,
- the individual net premium: P (assumption: uniform in the population).

The total cost is given by:

$$Total\ costs = n * \bar{c}$$

So

$$\bar{c} = \frac{C}{n}$$

Moreover the total received premiums are:

$$Total\ premiums = N * P$$

And the pricing principle is:

Equation 11: Pricing Principle

$$Total\ premiums = Total\ costs$$

3.2 Premiums

3.2.1 Premium without deductible

From the pricing principle we get,

$$N * P = n * \bar{c}$$

So

Equation 12: Average Cost Premium

$$P = \frac{n}{N} * \bar{c}$$

Where $\frac{n}{N}$ is the frequency of the claims.

Note 1: By using the expression $= n * \bar{c} * \frac{1}{N} = Total\ costs * \frac{1}{N}$, we understand that the total costs are shared between all individuals.

Note 2: In Practice this net premium is loaded with some fees by the company and interests are taken into account in the premium principle.

3.2.2 Premium with a deductible

Let us consider one insured ($N = 1$) with:

- the deductible: f,
- the claim amount: y,
- the present value of the cost of the claim: s.

We have:

$$s = \begin{cases} 0, & if\ y < f \\ y - f, & if\ y \geq f \end{cases}$$

From the premium principle, we have:

$$P = frequency * average\ cost = n * \bar{c}$$

We note F, the cumulative distribution of the claim amount. Then,

Without deductible we have:

$$\bar{c} = \int_0^{\infty} y\, dF(y)\ \ with\ \ F(y) = P(s < y)$$

With a deductible we have:

$$\bar{c}(f) = \int_0^\infty z\, dG(z) \;\; with \;\; z = y - f$$

And

$$G(z) = P(s < z + f \mid s > f)$$
$$= \frac{P(s < z + f \; and \; s > f)}{P(s > f)}$$
$$= \frac{F(z + f) - F(f)}{1 - F(f)}$$

So

$$dG(z) = dF(z + f)/\big(1 - F(f)\big) = dF(y)/\big(1 - F(f)\big)$$

And

$$\bar{c}(f) = \int_f^\infty (y - f) \;\, dF(y)/\big(1 - F(f)\big)$$

Hence, the net premium with the deductible f is given by:

$$P_f = (number \; of \; claims > f) * \bar{c}(f)$$
$$= n * P(s > f) * \bar{c}(f)$$
$$= n * \big(1 - F(f)\big) * \int_f^\infty (y - f) \;\, dF(y)/\big(1 - F(f)\big)$$

So

$$P_f = n * \int_f^\infty (y - f)\, dF(y)$$

When the net premium without the deductible is given by:

$$P = n * \int_0^\infty y\, dF(y)$$

So, the relative difference due to the deductible is given by:

$$P_f/P = \int_f^\infty (y - f)\, dF(y) \Big/ \int_0^\infty y\, dF(y)$$

The relative difference doesn't depend on the frequency, i.e. of the total numbers of claims. So, theoretically, the deductible allows the insurer to provide a discounted premium and reduce the moral hazard.

We have also:

$$P_f/P = 1 - \left[\int_0^f y\, dF(y) \Big/ \int_0^\infty y\, dF(y)\right] - \left[f * \big(1 - F(f)\big) \Big/ \int_0^\infty y\, dF(y)\right]$$
$$P_f/P = 1 - [relative \; weight \; of \; claims < f] - [relative \; discount \; of \; claims > f]$$

3.2.3 Statistical Models

In practice, the key points are the estimation (with statistical models), a priori, of:

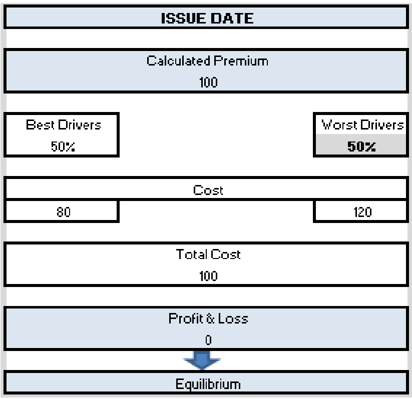
- the frequency of the claims for an insured,
- the cost of the claims.

However, there is often asymmetric information between the insurer and the insured. So, the statistical models have to take into account:

- the adverse selection and,
- the moral hazard.

To reduce the adverse selection, the companies could use a classification, a priori, of the risks from some observable variables like age, sex, work, type of car, space of living, first year of driving, number of drivers allowed. Then, they do some statistical studies from these variables. For example: apply the mean square estimation of the number of claims from these observable variables.

<u>Example: Adverse Selection:</u>



Despite of the equilibrium at time 0, the company could go to bankrupt because of adverse selection.

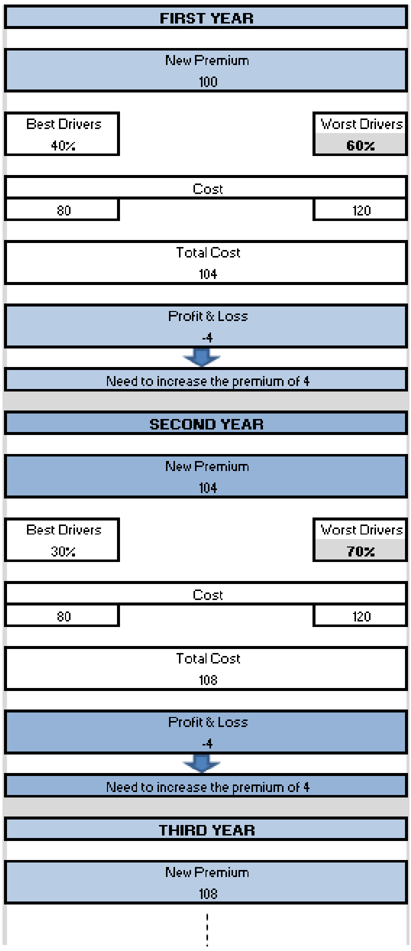


Figure 4: Adverse Selection

To reduce the moral hazard, the companies use the deductible and take into account the past experience of the insured (no claims bonus…).

3.3 Reserves

3.3.1 Background

An insurance policy provides, in return for the payment of a premium, acceptance of the liability to make payments to the insured person on the occurrence of one or more specified events (insurance claims) over a specific time period. The occurrence of the specified events and the amount of the payment are both usually modeled as random variables. In general, there is a delay in the insurer's settlement of the claim, typical reasons are reporting delay (time gap between claims occurrence and claims reporting at the insurance company); settlement delay because it usually takes time to evaluate the whole size of the claim. The time difference between claims occurrence and claims closing (final settlement) can be important.

Underwriting risk in non-life insurance is typically divided into premium risk and reserve risk. While premium risk deals with future claims, reserve risk considers claims that have already occurred. Reserve risk (or **claims reserves or outstanding claims**) focuses on uncertainty about future payments due to a claims settlement process and is one of the major types of risk for a non-life insurance company. Quantifying reserve risk thus plays an essential role in internal risk modeling of non-life insurance companies.

These reserves are set aside subject to the Enforcement Regulation of the Insurance Business Law, the statement showing the basis of working out premiums and underwriting reserves.

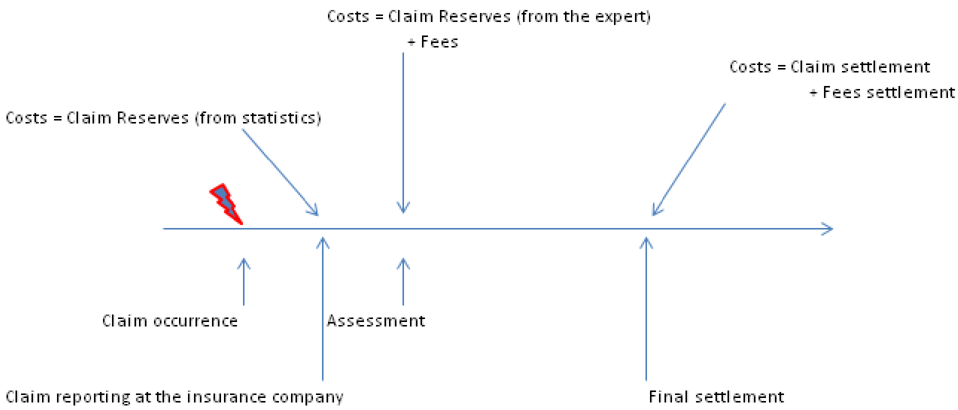


Figure 5: Claims Reserves

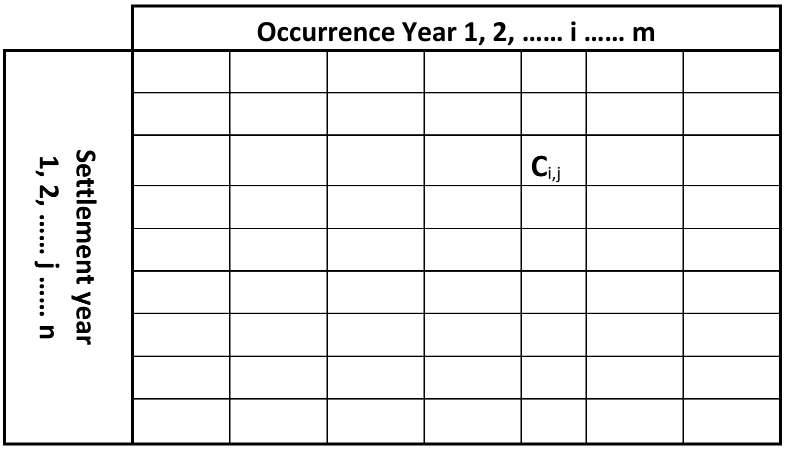
3.3.2 Chain Ladder method

There are several claims reserving methods that can be determinist or stochastic. A well-known statistical method of estimating outstanding claims is named Chain Ladder: the weighted average of past claim development is projected into the future. The projection is based on the ratios of cumulative past claims, usually paid or incurred, for successive years of development. It requires the earliest year of origin to be fully run-off or at least that the final outcome for that year can be estimated with confidence. If appropriate, the method can be applied to past claims data that have been explicitly adjusted for past inflation.

Let,

$\boldsymbol{C_{i,j}}$ = settlements at j for the claims occurred at i with $i \in [1, m]$ and $j \in [1, n]$,
$\boldsymbol{X_{i,j}}$ = Accrued settlements for the occurrence year i,

$$X_{i,j} = \sum_{k=1}^{j} C_{i,k}$$



Assumptions:

- Homogeneous data
- $t_j = X_{i,j+1}/X_{i,j}$ doesn't depend on i.

As an estimation of t_j is given by:

$$T_j = \frac{\sum_{k=1}^{m} X_{k,j+1}}{\sum_{k=1}^{m} X_{k,j}}$$

Then the estimated settlements $\boldsymbol{C_{i,j}}$ and the outstanding claims reserves can be calculated from the observed settlements.

Outstanding Claims Reserves - Chain Ladder

	Occurrence				
CLAIMS	1995	1996	19997	1998	1999
Claims paid in 1995	290				
Claims paid in 1996	184	300			
Claims paid in 1997	6	188	289		
Claims paid in 1998	3	11	177	287	
Claims paid in 1999	1	2	10	177	294
Claims paid in 2000	0	?	?	?	?
Claims paid in 2001	0	?	?	?	?
Claims paid in 2002	0	?	?	?	?
Claims paid in 2003	0	?	?	?	?
Final cost at 31/12/1999	?	?	?	?	?
Outstanding Claims Reserves	?	?	?	?	?

	Occurrence				
CLAIMS	1995	1996	19997	1998	1999
Accrued claims in 1995	290				
Accrued claims in 1996	474	300			
Accrued claims in 1997	480	488	289		
Accrued claims in 1998	483	499	466	287	
Accrued claims in 1999	484	501	476	464	294
Accrued claims in 2000	484				
Accrued claims in 2001	484				
Accrued claims in 2002	484				
Accrued claims in 2003	484				
Final cost at 31/12/1999	484				
Outstanding Claims Reserves	0				

j	T_j	
1	1,623	n → n+1
2	1,019	n+1 → n+2
3	1,005	n+2 → n+3
4	1,002	n+3 → n+4
5	1	n+4 → n+5

Assumption: claims of generation 1995 are closed at the end of 1999

	Occurrence				
CLAIMS	1995	1996	19997	1998	1999
Claims paid in 1995	290				
Claims paid in 1996	184	300			
Claims paid in 1997	6	188	289		
Claims paid in 1998	3	11	177	287	
Claims paid in 1999	1	2	10	177	294
Claims paid in 2000	0	1	2	9	183
Claims paid in 2001	0	0	1	2	9
Claims paid in 2002	0	0	0	1	2
Claims paid in 2003	0	0	0	0	1
Final cost at 31/12/1999	484	502	479	476	490
Outstanding Claims Reserves	0	1	3	12	196

	Occurrence				
CLAIMS	1995	1996	19997	1998	1999
Accrued claims in 1995	290				
Accrued claims in 1996	474	300			
Accrued claims in 1997	480	488	289		
Accrued claims in 1998	483	499	466	287	
Accrued claims in 1999	484	501	476	464	294
Accrued claims in 2000	484	502	478	473	477
Accrued claims in 2001	484	502	479	475	486
Accrued claims in 2002	484	502	479	476	489
Accrued claims in 2003	484	502	479	476	490
Final cost at 31/12/1999	484	502	479	476	490
Outstanding Claims Reserves	0	1	3	12	196

Figure 6: Chain Ladder

Remarks:

- This method implied a regular rhythm of the settlements,
- The estimations of the last year are not robust because they are calculated from only one data,
- Chain Ladder is adapted to long term claims (auto…).

References

Petauton, *Théorie et pratique de l'assurance vie*, 3° édition, Dunod.

Petauton, *Théorie de l'assurance dommages*. *Manuel et exercices corrigés*, 4ème édition, Dunod.

Vladimir I. Rotar, *Actuarial models: the mathematics of insurance*, CRC Press, 2006.

Thomas Mikosch, *Non-life insurance mathematics: an introduction with stochastic processes*, Volume 13, Springer, 2004.

http://en.wikipedia.org/, Wikipedia.

http://www.legifrance.gouv.fr/, Law.

Appendix A. Actuarial formulae

- **Probabilities on one insured**

	Survival probability at a given date		Death probability before a given age	
Probability of a life age x surviving to age x + 1	$p_x = \frac{l_{x+1}}{l_x}$	Death between ages x and x + 1	$q_x = \frac{l_x - l_{x+1}}{l_x} = \frac{d_x}{l_x}$	$p_x + q_x = 1$
Probability of a life age x surviving to age x + n	${}_np_x = \frac{l_{x+n}}{l_x}$	Death between ages x and x + n	${}_nq_x = \frac{{}_nd_x}{l_x} = \frac{l_x - l_{x+n}}{l_x}$	${}_np_x + {}_nq_x = 1$
ω maximum age	$p_\omega = 0$		$q_\omega = 1$	

- **Properties**

Probability of a life after n years ↔ Alive from x to x+1 and from x+1 to x+2...and from x+n-1 to x+n	${}_np_x = p_x * p_{x+1} * \dots * p_{x+n-1}$
Probability of a life after m+n years ↔ Alive from x to x+m and from x+m to x+m+n	${}_{m+n}p_x = {}_mp_x * {}_np_{x+m}$
Alive until m then die the following year i.e. before age x+m+1	${}_{m\|}q_x = {}_mp_x * q_{x+m}$
Alive until m, die before n years i.e. before age x+m+n	${}_{m\|n}q_x = {}_mp_x * {}_nq_{x+m}$
Alive until m but not in m+1	${}_{m\|}q_x = {}_mp_x - {}_{m+1}p_x$
Alive until m but not in m+n	${}_{m\|n}q_x = {}_mp_x - {}_{m+n}p_x$

- **Commutation symbols in case of life**

$D_x = v^x * l_x$
$N_x = \sum_{k=0}^{\omega-x} D_{x+k}$
$S_x = \sum_{k=0}^{\omega-x} N_{x+k}$

Pure Endowment:

$${}_{n}E_x = \frac{D_{x+n}}{D_x}$$

- **Life Annuities**

Life Annuities due at the end of the year	
Up to the death	$a_x = \frac{N_{x+1}}{D_x}$
Temporary	$a_{x:n} = \frac{N_{x+1} - N_{x+n+1}}{D_x}$
Deferred	${}_{m}a_x = \frac{N_{x+m+1}}{D_x}$
Temporary (n) and deferred (m)	${}_{m}a_{x:n} = \frac{N_{x+m+1} - N_{x+m+n+1}}{D_x}$
Properties	
Temporary (n) and deferred (m) = Temporary (n) at x+m discounted with ${}_{m}E_x$.	${}_{m}a_{x:n} = {}_{m}E_x * a_{x+m:n}$
Whole Life = Temporary (n) and deferred (n) Whole Life	$a_x = {}_{n}a_x + a_{x:n}$

Life Annuities due at the beginning of the year	
Up to the death	$\ddot{a}_x = \frac{N_x}{D_x}$
Temporary	$\ddot{a}_{x:n} = \frac{N_x - N_{x+n}}{D_x}$
Deferred	${}_{m}\ddot{a}_x = \frac{N_{x+m}}{D_x}$
Temporary (n) and deferred (m)	${}_{m}\ddot{a}_{x:n} = \frac{N_{x+m} - N_{x+m+n}}{D_x}$
Properties	
Whole Life = Temporary (n) and deferred (n) Whole Life	$\ddot{a}_x = {}_{n}\ddot{a}_x + \ddot{a}_{x:n}$

Life Annuities with several payment each year, due at the end of the period	
Up to the death	$a_x^{(k)} = a_x + \frac{(k-1)}{2*k}$
Temporary	$a_{x:n}^{(k)} = a_{x:n} + \frac{(k-1)}{2*k} * (1 - {}_{n}E_x)$
Deferred	${}_{m}a_x^{(k)} = {}_{m}a_x + \frac{(k-1)}{2*k} * {}_{m}E_x$
Temporary (n) and deferred (m)	${}_{m}a_{x:n}^{(k)} = {}_{m}E_x * a_{x+m:n}^{(k)}$

Life Annuities with several payment each year, due at the beginning of the period	
Up to the death	$\ddot{a}_x^{(k)} = \ddot{a}_x - \frac{(k-1)}{2*k}$
Temporary	$\ddot{a}_{x:n}^{(k)} = \ddot{a}_{x:n} - \frac{(k-1)}{2*k} * (1 - {}_{n}E_x)$
Deferred	${}_{m}\ddot{a}_x^{(k)} = {}_{m}\ddot{a}_x - \frac{(k-1)}{2*k} * {}_{m}E_x$
Temporary (n) and deferred (m)	${}_{m}\ddot{a}_{x:n}^{(k)} = {}_{m}E_x * \ddot{a}_{x+m:n}^{(k)}$

- **Commutation symbols in case of death**

<u>Assumption</u>: death at the end of the year.

$C_x = v^{x+1} * d_x$
$M_x = \sum_{k=0}^{\omega-x} C_{x+k}$
$R_x = \sum_{k=0}^{\omega-x} M_{x+k}$

- **Insurance in case of Death**

Whole Life	$A_x = \sum_{k=0}^{\omega-x-1} v^{k+1} * \frac{d_{x+k}}{l_x}$	$\frac{M_x}{D_x}$
Term Insurance	${}_n^{\blacksquare}A_x = \sum_{k=0}^{n-1} v^{k+1} * {}_{k\|1}^{\blacksquare}q_x$	$\frac{M_x - M_{x+n}}{D_x}$
Term Insurance, differed (m)	${}_{m\|n}^{\blacksquare}A_x = {}_m^{\blacksquare}E_x * {}_n^{\blacksquare}A_{x+m}$	$\frac{M_{x+m} - M_{x+m+n}}{D_x}$
Whole Life, differed (m)	${}_{m\|}^{\blacksquare}A_x = {}_m^{\blacksquare}E_x * A_{x+m}$	$\frac{M_{x+m}}{D_x}$
Term Insurance with increasing life annuities 1€, 2€, …, n€	${}_n^{\blacksquare}(IA)_x = \sum_{k=0}^{n-1} (k+1) * {}_{k\|1}^{\blacksquare}A_x$ $= \sum_{k=0}^{n-1} (k+1) * {}_{k\|1}^{\blacksquare}q_x * v^{k+1}$	$\frac{R_x - R_{x+n} - n * M_{x+n}}{D_x}$
Term Insurance with decreasing life annuities n€, (n-1)€, …, 1€	${}_n^{\blacksquare}(DA)_x = \sum_{k=0}^{n-1} (n-k) * {}_{k\|1}^{\blacksquare}A_x$	$\frac{n * M_x - (R_{x+1} - R_{x+n+1})}{D_x}$

- **Insurance in case of Life and Death**

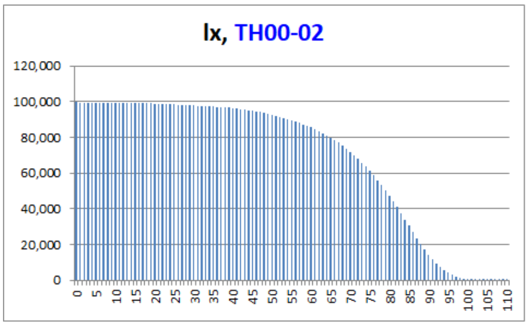
<u>Notation</u>: ***C***. = Benefit

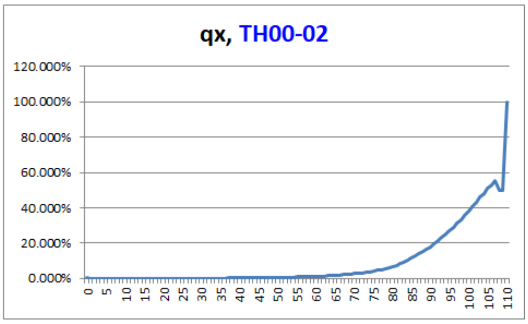
Endowment	$A_{x:\overline{n}\|} = C * {}_n^{\square}E_x + C * {}_n^{\square}A_x$	$\frac{C * \left(D_{x+n} + (M_x - M_{x+n})\right)}{D_x}$
Combined Endowment	$C_1 * {}_n^{\square}E_x + C_2 * {}_n^{\square}A_x$	$\frac{C_1 * D_{x+n} + C_2 * (M_x - M_{x+n})}{D_x}$

Appendix B. Mortality Tables

Mortality Table, France, TH00-02

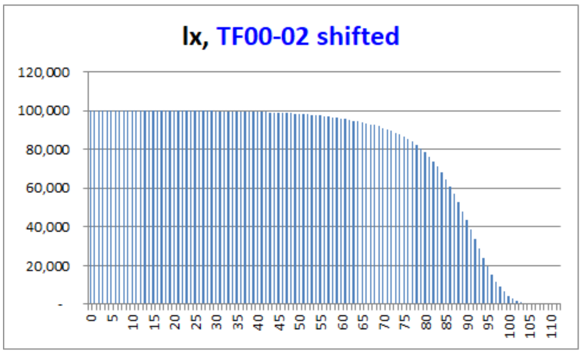
Age	Number surviving to age x	Probability of dying between ages x to x+1	Number dying between ages x to x+1
x	lx	qx	dx
0	100,000	0.489%	489
1	99,511	0.038%	38
2	99,473	0.027%	27
3	99,446	0.022%	22
4	99,424	0.018%	18
5	99,406	0.016%	16
6	99,390	0.014%	14
7	99,376	0.013%	13
8	99,363	0.013%	13
9	99,350	0.012%	12
10	99,338	0.013%	13
11	99,325	0.013%	13
12	99,312	0.016%	16
13	99,296	0.020%	20
14	99,276	0.026%	26
15	99,250	0.037%	37
16	99,213	0.050%	50
17	99,163	0.067%	66
18	99,097	0.083%	82
19	99,015	0.095%	94
20	98,921	0.102%	101
21	98,820	0.105%	104
22	98,716	0.105%	104
23	98,612	0.104%	103
24	98,509	0.105%	103
25	98,406	0.105%	103
26	98,303	0.107%	105
27	98,198	0.109%	107
28	98,091	0.111%	109
29	97,982	0.114%	112
30	97,870	0.116%	114
31	97,756	0.120%	117
32	97,639	0.125%	122
33	97,517	0.132%	129
34	97,388	0.143%	139
35	97,249	0.153%	149
36	97,100	0.166%	161
37	96,939	0.179%	174
38	96,765	0.195%	189
39	96,576	0.214%	207
40	96,369	0.237%	228
41	96,141	0.264%	254
42	95,887	0.293%	281
43	95,606	0.325%	311
44	95,295	0.360%	343
45	94,952	0.397%	377
46	94,575	0.435%	411
47	94,164	0.472%	444
48	93,720	0.508%	476
49	93,244	0.545%	508
50	92,736	0.582%	540
51	92,196	0.624%	575
52	91,621	0.668%	612
53	91,009	0.715%	651
54	90,358	0.767%	693
55	89,665	0.821%	736
56	88,929	0.875%	778
57	88,151	0.932%	822
58	87,329	0.995%	869
59	86,460	1.066%	922
60	85,538	1.146%	980
61	84,558	1.235%	1,044
62	83,514	1.335%	1,115
63	82,399	1.448%	1,193
64	81,206	1.576%	1,280
65	79,926	1.719%	1,374
66	78,552	1.876%	1,474
67	77,078	2.046%	1,577
68	75,501	2.232%	1,685
69	73,816	2.434%	1,797
70	72,019	2.658%	1,914
71	70,105	2.903%	2,035
72	68,070	3.167%	2,156
73	65,914	3.455%	2,277
74	63,637	3.768%	2,398
75	61,239	4.117%	2,521
76	58,718	4.506%	2,646
77	56,072	4.938%	2,769
78	53,303	5.426%	2,892
79	50,411	5.993%	3,021
80	47,390	6.660%	3,156
81	44,234	7.433%	3,288
82	40,946	8.304%	3,400
83	37,546	9.253%	3,474
84	34,072	10.264%	3,497
85	30,575	11.352%	3,471
86	27,104	12.533%	3,397
87	23,707	13.802%	3,272
88	20,435	15.155%	3,097
89	17,338	16.576%	2,874
90	14,464	18.059%	2,612
91	11,852	19.625%	2,326
92	9,526	21.289%	2,028
93	7,498	23.059%	1,729
94	5,769	24.926%	1,438
95	4,331	26.899%	1,165
96	3,166	28.964%	917
97	2,249	31.125%	700
98	1549	33.376%	517
99	1032	35.756%	369
100	663	38.160%	253
101	410	40.488%	166
102	244	43.033%	105
103	139	46.043%	64
104	75	48.000%	36
105	39	51.282%	20
106	19	52.632%	10
107	9	55.556%	5
108	4	50.000%	2
109	2	50.000%	1
110	1	100.000%	1

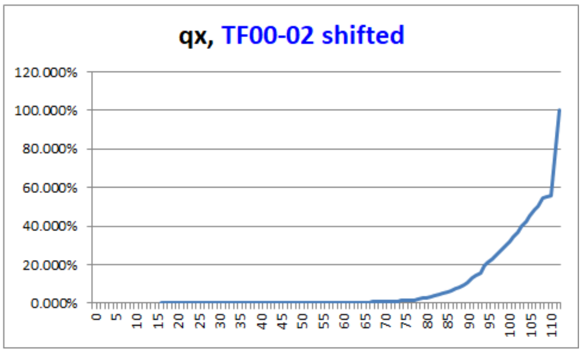




Mortality Table, France, TF00-02 shifted

Age	Number surviving to age x	Probability of dying between ages x to x+1	Number dying between ages x to x+1
x	lx	qx	dx
0	100,000		
1	100,000		
2	100,000		
3	100,000		
4	100,000		
5	100,000		
6	100,000		
7	100,000		
8	100,000		
9	100,000		
10	100,000		
11	100,000		
12	100,000		
13	100,000		
14	100,000		
15	100,000		
16	100,000	0.012%	12
17	99,988	0.011%	11
18	99,977	0.010%	10
19	99,967	0.010%	10
20	99,957	0.010%	10
21	99,947	0.011%	11
22	99,936	0.011%	11
23	99,925	0.012%	12
24	99,913	0.013%	13
25	99,900	0.016%	16
26	99,883	0.020%	20
27	99,863	0.024%	24
28	99,839	0.029%	29
29	99,810	0.033%	33
30	99,777	0.035%	35
31	99,742	0.035%	35
32	99,707	0.034%	34
33	99,672	0.035%	35
34	99,637	0.035%	35
35	99,602	0.039%	39
36	99,563	0.042%	42
37	99,521	0.047%	46
38	99,474	0.052%	51
39	99,423	0.058%	57
40	99,366	0.064%	63
41	99,302	0.070%	69
42	99,233	0.076%	75
43	99,157	0.084%	84
44	99,074	0.093%	93
45	98,981	0.103%	102
46	98,879	0.114%	113
47	98,767	0.125%	124
48	98,643	0.138%	136
49	98,507	0.151%	149
50	98,358	0.165%	162
51	98,196	0.195%	192
52	98,004	0.210%	206
53	97,799	0.224%	219
54	97,580	0.238%	232
55	97,348	0.251%	244
56	97,104	0.265%	257
57	96,847	0.282%	273
58	96,574	0.302%	291
59	96,282	0.324%	312
60	95,971	0.346%	332
61	95,639	0.392%	374
62	95,264	0.414%	395
63	94,870	0.440%	417
64	94,453	0.468%	442
65	94,011	0.503%	473
66	93,538	0.543%	508
67	93,030	0.589%	548
68	92,482	0.697%	645
69	91,838	0.763%	701
70	91,137	0.838%	764
71	90,373	0.923%	834
72	89,539	1.020%	914
73	88,625	1.133%	1,004
74	87,621	1.259%	1,103
75	86,518	1.401%	1,212
76	85,306	1.563%	1,333
77	83,972	1.752%	1,471
78	82,501	2.228%	1,838
79	80,663	2.527%	2,039
80	78,625	2.874%	2,260
81	76,365	3.276%	2,502
82	73,863	3.749%	2,769
83	71,094	4.304%	3,060
84	68,034	4.956%	3,372
85	64,662	5.709%	3,692
86	60,971	6.564%	4,002
87	56,968	7.518%	4,283
88	52,685	8.567%	4,514
89	48,172	9.693%	4,669
90	43,502	10.865%	4,727
91	38,776	13.260%	5,142
92	33,634	14.546%	4,892
93	28,742	15.965%	4,589
94	24,153	19.229%	4,644
95	19,509	21.060%	4,109
96	15,400	23.005%	3,543
97	11,857	25.045%	2,970
98	8,888	27.208%	2,418
99	6,470	29.473%	1,907
100	4,563	31.837%	1,453
101	3,110	34.316%	1,067
102	2,043	36.886%	754
103	1,289	39.556%	510
104	779	42.279%	329
105	450	45.223%	203
106	246	48.256%	119
107	127	50.562%	64
108	63	54.545%	34
109	29	55.000%	16
110	13	55.556%	7
111	6	75.000%	4
112	1	100.000%	1





Appendix C. Actuary

- **French actuarial organization site: http://www.institutdesactuaires.com**

Institut des Actuaires

Actuaire®
CALCULER L'AVENIR

Tableau Unique des Actuaires

A B C D E F G H I J K L M
N O P Q R S T U V W X Y Z

	NOM	PRENOM	PROMO
			2004
			1993
			2010

Tableau des Membres d'Honneur

A B C D E F G H I J K L M
N O P Q R S T U V W X Y Z

	NOM	PRENOM	PROMO
			1986
			1988

- **Actuarial studies in France**

- **Responsibilities**

Actuaries use skills in mathematics, economics, computer science, finance, probability and statistics, and business to help businesses assess the risk of certain events occurring and to formulate policies that minimize the cost of that risk. For this reason, actuaries are essential to the insurance and reinsurance industry, either as staff employees or as consultants; to other businesses, including sponsors of pension plans; and to government agencies such as the Government Actuary's Department in the UK or the Social Security Administration in the US. Actuaries assemble and analyze data to estimate the probability and likely cost of the occurrence of an event such as death, sickness, injury, disability, or loss of property. Actuaries also address financial questions, including those involving the level of pension contributions required to produce a certain retirement income and the way in which a company should invest resources to maximize its return on investments in light of potential risk. Using their broad knowledge, actuaries help design and price insurance policies, pension plans, and other financial strategies in a manner which will help ensure that the plans are maintained on a sound financial basis.

- **Traditional employment**

On both the life and casualty sides, the classical function of actuaries is to calculate premiums and reserves for insurance policies covering various risks. Premiums are the amount of money the insurer needs to collect from the policyholder in order to cover the expected losses, expenses, and a provision for profit. Reserves are provisions for future liabilities and indicate how much money should be set aside now to reasonably provide for future payouts. If you inspect the balance sheet of an insurance company, you will find that the liability side consists mainly of reserves.

On the casualty side, this analysis often involves quantifying the probability of a loss event, called the frequency, and the size of that loss event, called the severity. Further, the amount of time that occurs before the loss event is also important, as the insurer will not have to pay anything until after the event has occurred. On the life side, the analysis often involves quantifying how much a potential sum of money or a financial liability will be worth at different points in the future. Since neither of these kinds of analysis are purely deterministic processes, stochastic models are often used to determine frequency and severity distributions and the parameters of these distributions. Forecasting interest yields and currency movements also plays a role in determining future costs, especially on the life side.

Actuaries do not always attempt to predict aggregate future events. Often, their work may relate to determining the cost of financial liabilities that have already occurred, called retrospective reinsurance, or the development or re-pricing of new products.

Actuaries also design and maintain products and systems. They are involved in financial reporting of companies' assets and liabilities. They must communicate complex concepts to clients who may not share their language or depth of knowledge. Actuaries work under a strict code of ethics that covers their communications and work products, but their clients may not adhere to those same standards when interpreting the data or using it within different kinds of businesses.

- **Non-traditional employment**

Many actuaries are general business managers or financial officers. They analyze business prospects with their financial skills in valuing or discounting risky future cash flows, and many apply their pricing expertise from insurance to other lines of business. Some actuaries act as expert witnesses by applying their analysis in court trials to estimate the economic value of losses such as lost profits or lost wages.

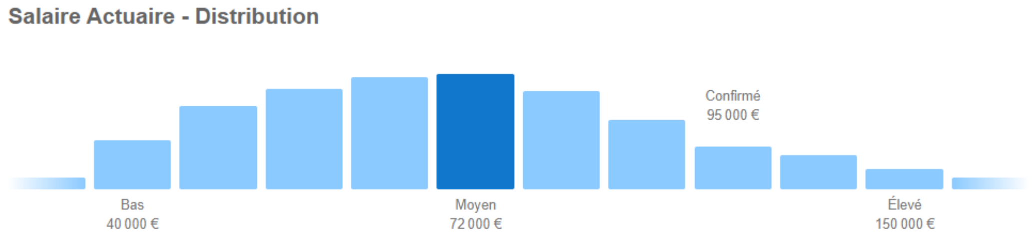
There has been a recent widening of the scope of the actuarial field to include investment advice and asset management. Further, there has been a convergence from the financial fields of risk management and quantitative analysis with actuarial science. Now, actuaries also work as risk managers, quantitative analysts, or investment specialists. Even actuaries in traditional roles are now studying and using the tools and data previously in the domain of finance. One of the latest developments in the industry, insurance securitization, requires both the actuarial and finance skills.

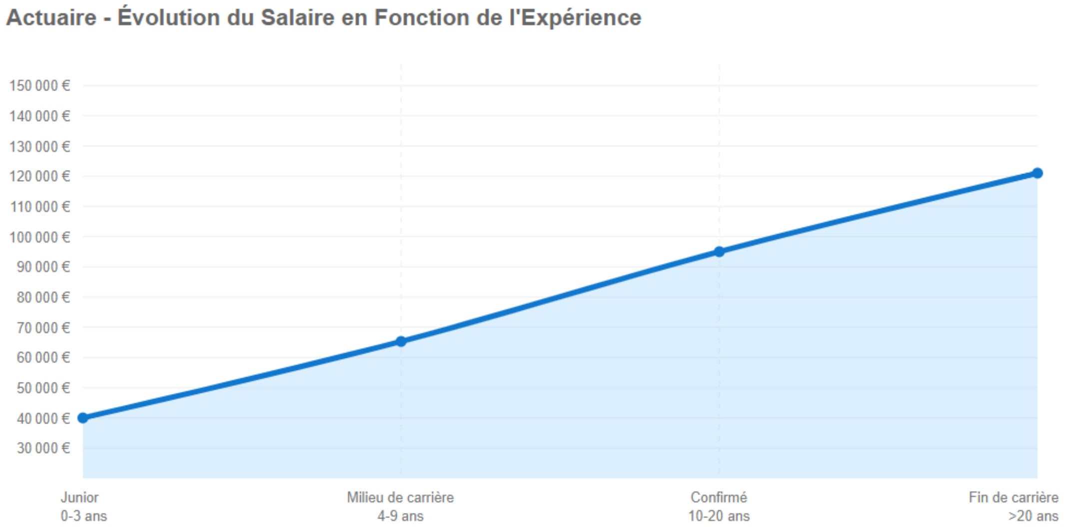
Another field in which actuaries are becoming more prominent is that of Enterprise Risk Management, for both financial and non-financial corporations. For example, the Basel II accord for financial institutions, and its analogue, the Solvency II accord for insurance companies, requires such institutions to account for operational risk separately and in addition to credit, reserve, asset, and insolvency risk. Actuarial skills are well suited to this environment because of their training in analyzing various forms of risk, and judging the potential for upside gain, as well as downside loss associated with these forms of risk.

- **Salary**

The credentialing and examination procedure for becoming a fully qualified actuary can be intensely demanding. Consequently, the profession remains small throughout the world. As a result, actuaries are in high demand, and they are highly paid for the services they render.

In France we have:





Source : https://fr.jobted.com/salaire/actuaire